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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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THE PRESIDENCY COLLEGE,
MADRAS,
18*th February* 1926.

A. NARASINGA RAO,
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we see that

$$H\zeta_n << \frac{\sum_{p=0}^{nm} |D_{n,p}| \cos^p \psi}{[H(\rho\psi)]^n}.$$

where $\quad |D_{n,p}| < Lz^p . H_{c(3nm-8),p}$.

Again $\quad \zeta_n = H\zeta_n \dfrac{1}{H(\rho\psi)}$ and therefore

$$\zeta_n << \frac{\sum_{p=0}^{nm} \left|F_{n,p}\right| \cos^p \psi}{[H(\rho\psi)]^n}; \ |P_{n,p}| < Lz^p H_{c(3nm-7),p}. \quad (44.6)$$

Hence we have shown that if we assume that $\zeta_1 \zeta_2 \ldots\ldots \zeta_{n-1}$ satisfy the conditions of the form (43.2), (43.3) then the next set of functions, *viz* W_n, $H\zeta_n$ and ζ_n will also satisfy similar conditions. Therefore we need only verify that these hold for any special value of n. But for $n = 1$ this is obvious from the form of ζ_1. Hence the functions ζ_n all satisfy conditions of the form (43.2), (43.3).

Further from these conditions, we have at once

$$\zeta_n < \frac{L}{[H(\rho\psi)]^n} \frac{1}{(1-z)^{c(3nm-7)}};$$

or $$|\zeta_n| < L(1-z)^{7c} \frac{1}{\rho^n.(1-z)^{3cnm}}. \quad (44.7)$$

We have thus shown that in the case when the angular velocity of the Pear-shaped figure is the same as that of the Elliptic cylinder from which it is derived (so that η does not occur in our equations) there exist numbers l_n such that $|\zeta_n| < l_n$. Further the form of the numbers l_n shows that for sufficiently small values of α, $\sum_{n>0} (l_n \alpha^n)$ is convergent. Thus the assumptions we made in the preceding parts are fully justified.

§ 45. In exactly the same way, when the figure of comparison is the fixed cylinder given by the equation of the point of Bifurcation and whose angular velocity is given by $\Omega_0 = \dfrac{\omega_0^2}{2}$ while that of the

Pear-shaped figure is given by $\Omega_0 + \eta = \frac{\omega^2}{2}$, we can show that the series

$$\zeta = \sum_{r+s>0} (\zeta_{rs}\, \alpha^r \eta^s) \qquad (45.1)$$

is absolutely and uniformly convergent, that is to say, there exist numbers l_{rs} such that $| \zeta_{rs} | < l_{rs}$ and that, for sufficiently small values of α, η, the series

$$\sum_{r+s>0} (l_{rs}\, \alpha^r \eta^s)$$

is convergent. Also remembering that the equation

$$0 = A \equiv A_2\, \alpha^2 + A_3 \alpha^3 + \ldots . + (B + \Sigma)\, \alpha\eta + C_3 \eta^3 + \ldots \qquad (45.2)$$

has in the case of non-elliptical figures of equilibrium to be solved for η in terms of α, the final expression for ζ is got by substituting this value of η [obtained from (45.2)] in the series (45.1) for ζ and rearranging it in terms of α. This is legitimate provided α be less than a certain limit, or, in other words; if η is less than a certain limit.

Thus by making η sufficiently small we can always define a set of Pear-shaped figures near the given Elliptic cylinder, which are in unstable equilibrium.

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Elementary Mathematics in 1700

By G. A. Miller.

In order to obtain a clear notion of the history of mathematics, it is very useful to fix the attention occasionally on a definite date and to consider some of the fundamental facts of our subject which were then either definitely known or definitely unknown. In the present article, we have selected 1700 as such a date, and we shall confine our attention to the history of Elementary Mathematics. It should be added that the choice of this date and the limitation to elementary mathematics were directly due to the fact that we hoped that we might thereby also throw some light incidentally on the opening sentence of Chapter X, Volume 1, of D. E. Smith's recent "*History of Mathematics*," which reads as follows: "Since this work is concerned primarily with the history of elementary mathematics it would be quite justifiable to set its limit at the close of the 17th century." A somewhat similar view is expressed in the article on mathematics found in the "New International Encylopædia," Volume 15 (1923), page 238.

The term elementary mathematics is used here with the meaning that it includes all the mathematics which is commonly taught to students until after they have completed their first course in the differential and the integral calculus. A study of the question how much of this mathematics was unknown in 1700 should be of special interest to the teachers of this subject, as well as to those students who have not proceeded beyond this stage in their mathematical work. Such a study may lead to many difficulties. For instance, when one reads in a favourably received history that the mathematics of the eighteenth century suffered because of the fact that a generally satisfactory introduction of the negative numbers was then missing,* one is led to the difficulty of determining to what extent those who used these numbers earlier understood their significance and their bearing, and the great differences as regards an understanding on the part of such users of them.

The subject of determinants presents much easier questions since so very little of this subject was known before 1700, and the little that was

* J. Tropfke: *Geschichte der Elementar-Mathematik*, Vol. 2 (1921), p. 78.

done before this date towards the development of this subject seems to have had little, if any, influence on its later development. Hence one may be inclined to agree with the statement that the real founder of determinants is Gabriel Cramer (1704—1752), which is found in Volume 1 (1910), page 52, of the well-known Pascal's "Repertorium der höheren Mathematik," and in many other places. On the other hand Thomas Muir regarded A. T. Vandermonde (1735—1796) as the founder of this subject.* The main fact that interests us at present is that this subject had its origin after the year 1700. In particular, the simple rule for expanding a determinant of the third order which appears in many college algebras and is commonly known as the rule of Sarrus was not published before the first half of the nineteenth century.

Even the technical term "determinant" with its modern meaning was first used more than a century after 1700 and the square form in which a determinant is now usually written began to be used about the same time. In view of the great simplifications which have been effected in the theory of the solution of a system of m linear equations in n unknowns,—for example, by the use of the notion of rank of a determinant—it is clear that the modern student of elementary mathematics finds in the theory of determinants a subject where he realizes the great importance of mathematical developments which were entirely unknown in 1700. Even some of the mathematical developments of the first half of the nineteenth century suffered because the subject of determinants had not then been sufficiently developed.

In order to understand the history of determinants. it is essential to have a clear notion of what is meant by the technical term "determinant." To most students who are familiar with only the elements of determinants this term probably suggests first the square array of elements which is really the matrix of the determinant. If this matrix with its modern meaning is regarded as an essential part of a determinant, the history of determinants begins long after 1700. On the other hand, if by determinant we mean the polynomial which this square array now suggests, then it must be admitted that the history of determinats begins with certain developments which antedate 1700. Among these developments one finds some due to G. W. Leibniz (1646—1716) which are

* "*Theory of Determinants in the Historical Order of Development,*" Volume 1 (1906), page 131.

especially significant. It should be noted, however, that not all the work of Leibniz which relates to determinants antedates 1700. In fact, in 1708 he gave a rule for expressing the roots of two equations in two unknowns in terms of the co-efficients, but in a letter written about thirty years earlier, he had already spoken of expressing in such a general way the roots of a system of linear equations.

It may be assumed that most of the students of the history of elementary mathematics are also interested in the development of the form in which elementary mathematics is now commonly presented in our text-books for beginners. This form is often so intimately associated with the subject matter itself that the beginner would frequently find it difficult to recognize the latter as developed in the earlier works along the same line. In particular, the work on determinants which antedates 1700 is very different in form from the developments of this subject in modern text-books. Those who really understand the subject find in this earlier work much that is of interest, but the fact that it is not known to have exerted any influence on later workers in this field tends to show that some of the really inspiring elements of the subject had been missed by these earlier workers. At any rate, a history of the elements of determinants should include advances in this subject which were made after the year 1700 and should devote most of its space to such advances.

Another important subject of elementary mathematics in which very definite progress since the year 1700 is easily recognized is furnished by the theory of ordinary complex numbers. It is obvious that our common general symbol $(a + bi)$ for such a number could not have been employed as early as the year 1700, since the use of i for $\sqrt{(-1)}$ is due to E. Euler (1707—1783). Moreover, from the fact that the first satisfactory geometric proof of the legitimacy of operating with these numbers, under certain restrictions, in the same manner as with real numbers, is commonly credited to Caspar Wessel (1745—1818), it is evident that our modern development of these numbers in the text-books on elementary mathematics must be largely based on work which was done ulterior to the year 1700. In particular, since imaginary exponents were first used by L. Euler in 1740 it is clear that such elegant formulas as

$$\sin x = \frac{e^{xi} - e^{-xi}}{2i}, \ \cos x = \frac{e^{xi} + e^{-xi}}{2}$$

which are now found in many of our text-books on elementary mathematics could not have been known before this date.

The subject of ordinary complex numbers was doubtless more extensively developed before the year 1700 than that of determinants. This development was largely inspired by the fact that real roots of a cubic equation often present themselves in the form of cube roots of complex numbers when these roots are found by the so-called formula of Cardan, as well as by the fact that by means of these numbers it became possible to say that an algebraic equation has as many roots as its degree. It is true that the latter fact was not definitely established until about the end of the eighteenth century; but as early as 1629, A. Girard noted that the four roots of the equation

$$x^4 - 4x + 3 = 0 \text{ are } 1,\ 1,\ -1 + \sqrt{-2},\ -1 - \sqrt{-2},$$

and that the product of the last two roots is 3 H. Cardan had observed incidentally at a much earlier date that

$$(5 + \sqrt{-15})\,(5 - \sqrt{-15}) = 40,$$

but the very useful transformation

$$a + bi = r\,(\cos\phi + i\sin\phi)$$

appeared for the first time a number of years after 1700 in the works of Euler and D'Alembert. Hence it is clear that the history of elementary mathematics as far as it relates to ordinary complex numbers should not close with the year 1700.

In elementary trigonometry, we find again very fundamental advances since the year 1700. It is now customary to think of the trigonometric functions of an angle primarily as abstract numbers even if the line definitions of these functions are also sometimes used. According to J. Tropfke, these functions were regarded explicitly as pure numbers for the first time in 1764;* but L. Euler seems to have used them with this meaning at a somewhat earlier date. While some mathematical historians have been inclined to credit earlier writers with he use of the ratio definitions of the trigonometric functions, it is obvious that only a little was done along this line before the year 1700, and it must be admitted that our most common modern definitions of the

* "*Geschichte der Elementar-Mathematik*," Volume 5 (1923), page 22.

trigonometric functions came into general use after this date. Hence we may say that the very beginning of trigonometry is concerned with developments made after the year 1700. This is the more interesting when we bear in mind the fact that the word *seqt* found in the ancient Egyptian work by Ahmes seems to imply that an approach towards the ratio definition of a trigonometric function was made even as early as 1700 B.C. and that an explicit ratio definition of such functions was still lacking in 1700. A.D.

In the elementary text-books on analytic geometry and differential and integral calculus, we naturally find a relatively larger number of results which became known after the year 1700. As a conspicuous example we may cite Taylor's Theorem which is due to B. Taylor (1685—1731) and appeared in a work published by him in 1715 but was announced by him in a letter as early as 1712. It attracted little attention until long after the death of its author when it was completed along important lines by J. L. Lagrange, A. L. Couchy, and others. On page XXIV of Volume 2 (1916) of the "*Ouvres de Henri Poincare,*" G. Darboux said that of all the discoveries which the mathematicians made in the course of the nineteenth century the most fertile without doubt is that which we owe to Couchy relative to Taylor's series and the conditions of its convergence. While the influence of Taylor's Theorem has been especially pronounced in more advanced fields of analysis, it is also quite marked in our elementary text-books on differential calculus.

Although R. Descartes called attention to the fact that his methods could be used in solid geometry, the subject of solid analytic geometry received very little attention until after the beginning of the year 1700. During this year, A. Parent, who has the honour of having published in printed form the first memoir on solid analytic geometry, prsented his researches before the Paris Academy. Even plane analytic geometry did not assume its modern form before the close of the eighteenth century. In particular, the modern forms of the equations of the tangent line of a circle, ellipse, parabola, and hyperbola did not appear before this time * It is somewhat singular that early writers paid very little attention to the analytic geometry of the straight line. In particular, the so-called

* J. Tropfke, "*Geschichte der Elementar-Mathematik,*" Volume 6 (1924), page 111.

Hessian normal form of the equation of a straight line, which is used so much in the modern text-books on elementary analytic geometry, seems to have been first employed by L'Huiller in 1809, and the well-known modern form of the equation of a line passing through two given points appeared for the first time about ten years earlier.

In view of the fact, noted above, that negative numbers were not yet generally understood in 1700, it is not surprising to find that the early writers on analytic geometry failed to treat the four quadrants as equals. One naturally finds great variations as regards their use of negative numbers, and the great simplifications resulting from the employment of these numbers in the development of analytic geometry doubtless contributed towards their more rapid adoption as real numbers during the eighteenth century. One of the most attractive features of the history of mathematics is such interplay between the development of various subjects. In particular, our modern analytic geometry would be impossible without the use of negative numbers and the introduction of the negative numbers hastened the development of analytic geometry. The fact that such a noted writer as Descartes called the negative numbers *false numbers* would naturally tend to make some later writers suspicious as regards the reality of these numbers, notwithstanding their apparent usefulness.

Partial fractions occupy a prominent position in the developments, of the elements of integral calculus and hence it is interesting to note here that this subject began to be developed by G. W. Leibniz and John Bernoulli very soon after the year 1700. In view of the fact that the fundamental partial fraction theorem of algebra was not yet established at that time, it is obvious that the treatment of the theory of these fractions could not then have been satisfactory. It is true that a proof of this theorem is not commonly given in our courses in elementary mathematics, but probably most teachers of the more advanced of such courses find it desirable to tell their students some things about the nature of the theorem and its bearing on some of the subjects which appear in these courses. Inasmuch as mathematical developments which are not elementary affect the treatment of elementary subjects, their history belongs to the history of elementary mathematics, and hence this history should not be too narrowly circumscribed. In particular, it should convey a clear view of the modern tendency towards the arithmetization of mathematics.

If one would place the lower and upper limits of elementary mathematics as low and as high respectively as is done in the well-known "*Encyklopadie der Elementar-Mathematik*" by Weber and Wellstein, one would find that the greater part of its developments were either entirely unknown in 1700, or were very imperfectly understood at that time. Moreover, if we compare the mathematical instruction at that time with the instruction at present, one finds a most encouraging improvement, especially as regards the number of young people who follow such instruction beyond the most elementary steps. In particular, it may be noted that in 1700, and for many years thereafter, elementary arithmetic was a senior subject in Harvard College, and it was more than a century after 1700 when elementary mathematical instruction in Germany passed from the University to the Gymnasium.*

The few facts noted above may suffice to furnish some idea of the actual state of elementary mathematics in the year 1700, and of the wonderful progress which has been made in this subject during the last two hundred and twenty-five years. It is a difficult matter to determine how much of the space of a general history of elementary mathematics should be devoted to this period; but, in view of the fact that it lies nearest to our own time and some of the developments started therein are still in progress, one might be inclined to think that many of the readers of such a history would be especially interested in this period even if the developments made therein usually relate to somewhat more difficult mathematical questions than those of earlier periods. At any rate, it should be of interest to teachers of elementary mathematics to note that definite progress relating to their subjects of instruction has been sufficiently recent to inspire the hope that the stationary stage has not yet been reached as regards their development. Progress in subject matter as well as in method should be an inspiring motive of teachers of elementary mathematics as well as of those whose work lies beyond this stage.

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* Cf. M. Gebhardt: "*Abhandlungen uber den mathematischen Unterricht in Deutschland.*" Volume 3, part 6 (1912), page 2.

On the application of Herr Mellin's integrals to some series

By S. L. Malurkar.

[The following paper is mainly the outcome of an attempt to prove the relation (1.014) which was proposed as a problem in J. I. M. S. by Mr. M. Bheemasena Rao. It was found possible by using Herr Mellin's integral to obtain a greater generalization of (1.014) in (1.017) for positive as well as negative values of m. Also it seemed to suggest an asymptotic series for small values of a when the value of μ was not an odd integer. This asymptotic series for $\mu = 2$ had been given by Mr. S. Ramanujan as cited in the obituary notice by Mr. G. H. Hardy.

The results of allied series are given in the other sections.

Lastly, the method is applied to derive Jacobi's 'Imaginary Transformation' from Hurwitz's for Riemann zeta function].

Starting from Herr Mellin's integral

$$e^{-y} = \frac{1}{2\pi i}\int_{(L)} \Gamma(s)\, y^{-s}\, ds$$

where the path of integration (L) extends from $(k - i\infty)$ to $(k + i\infty)$ along a straight line $\quad R(s) = R(k)$

and $R(y) > 0,\ R(k) > 0,$

it is easy to see that

$$\frac{1}{e^y - 1} = \sum_{r=1}^{\infty} e^{-ry} = \frac{1}{2\pi i}\int_{(L)} \Gamma(s) \sum_{1}^{\infty} (ry)^{-s}\, ds$$

which converges, to $\dfrac{1}{2\pi i}\displaystyle\int_{(L)} \Gamma(s)\,.\,\zeta(s)\,.\,y^{-s}\, ds,$

if $R(k-1) > 0,\ R(y) > 0.$

Hence using the transformation

$$2\Gamma(s)\zeta(s)\cos\tfrac{1}{2}\pi s = (2\pi)^s\,\zeta(1-s)$$

we have
$$\frac{1}{e^y - 1} = \frac{1}{2\pi i}\int_{(L)} \frac{\zeta(1-s)\, y^{-s}}{2\cos\frac{1}{2}\pi s}\,.\,(2\pi)^s\, ds. \qquad (1.00)$$

Similarly $\frac{1}{e^y - e^{-y}} = \sum_{r=0}^{\infty} e^{-(2r+1)y}$

$$= \frac{1}{2\pi i} \int_{(L)} \Gamma(s) \, . \sum_{0}^{\infty} (\overline{2r+1}\, y)^{-s} \, ds \, ;$$

$$= \frac{1}{2\pi i} \int_{(L} \Gamma(s) \, . \, \zeta(s) \, . \, (1-2^{-s}) \, y^{-s} \, ds. \; [\mathrm{R}(k-1) > 0$$

$$= \frac{1}{2\pi i} \int_{(L)} \frac{\zeta(1-s) \, . \, (1-2^{-s}) \, . \, (y/2\pi)^{-s} \, ds}{2 \cos \pi s/2} \, ; \quad (1.10)$$

and $\frac{1}{e^y + 1} = \sum_{r=1}^{\infty} (-)^{r-1} e^{-ry}$

$$= \frac{1}{2\pi i} \int_{(L)} \Gamma(s) \sum_{1}^{\infty} (-)^{r-1} (ry)^{-s} \, ds$$

$$= \frac{1}{2\pi i} \int_{(L)} \Gamma(s) \, \zeta(s) \, . \, (1-2^{1-s}) \, y^{-s} \, ds \quad [\mathrm{R}(k) > 0$$

$$= \frac{1}{2\pi i} \int_{(L)} \frac{\zeta(1-s) \, . \, (1-2^{1-s})}{2 \cos \frac{1}{2} \pi s} \, . \, (y/2\pi)^{-s} \, ds. \; (1.20)$$

Again $\frac{1}{e^y + e^{-y}} = \sum_{0}^{\infty} (-)^r e^{-(2r+1)y}$

$$= \frac{1}{2\pi i} \int_{(L)} \Gamma(s) \, . \sum_{r=0}^{\infty} (-)^r (2r+1)^{-s} \, y^{-s} \, ds$$

which converges to

$$\frac{1}{2\pi i} \int_{(L)} \Gamma(s) \, \eta(s) \, y^{-s} \, ds$$

where $\eta(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} \cdots$

and its continuations.

Using the transformation

$$\Gamma(s).\eta(s)\sin\frac{\pi s}{2} = \left(\frac{\pi}{2}\right)^s \eta(1-s),$$

$$\frac{1}{e^y + e^{-y}} = \frac{1}{2\pi i}\int_{(L)} \frac{\eta(1-s).(2y/\pi)^{-s}\,ds}{\sin \pi s/2}. \quad (1.30)$$

From (1.00) it follows that

$$\sum_{r=1}^{\infty} \frac{r^{\mu}}{e^{\pi a r}-1} = \frac{1}{2\pi i}\int_{(L)} \frac{\zeta(1-s).a^{-s}}{2\cos \pi s/2}\left\{\sum^{\infty} r^{\mu - s}\right\}ds.$$

The integrand converges if $R(k-\mu) > 1$ and $R(k) > 1$.

Hence

$$\sum_{1}^{\infty} \frac{r^{\mu}}{e^{2\pi a r}-1} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\zeta(1-s)\;\zeta(s-\mu).a^{-s}\,ds}{2\cos\pi s/2}. \quad (1.01)$$

$$= \frac{1}{2\pi i}\int_{1+\mu-k+i\infty}^{1+\mu-k-i\infty} \frac{\zeta(1-s)\,\zeta(s-\mu).a^{s-\mu-1}\,ds}{2\cos\frac{1}{2}\pi(s-\mu-1)} \quad (1.011)$$

by change of variable.

If μ is not an integer, it is possible to obtain an asymptotic formula for the integral (1.01) for sufficiently small values of a.

The contour integral can be replaced by one taken along a line $(-2m+1-\beta+it)$ together with the sum of the residues at the poles of the integrand lying between the lines. Hence

$$\frac{1}{2\pi i}\int_{(L)} \frac{\zeta(1-s)\,\zeta(s-\mu)\,a^{-s}\,ds}{2\cos\pi s/2}$$

$$= \frac{\zeta(-\mu)}{2}\left\{\frac{a^{-\mu-1}}{\cos\pi(\mu+1)/2} - 1\right\}$$

$$+ \sum_{r} \frac{a^{-2r-1}\,\zeta(-2r)\,\zeta(2r+1-\mu)}{-\pi\sin(2r+1)\,\pi/2}$$

$$+ \frac{1}{2\pi i}\int_{-2m+1-\beta-i\infty}^{-2m+1-\beta+i\infty} \frac{\zeta(1-s)\,\zeta(s-\mu)\,a^{-s}\,ds}{2\cos\pi s/2}, \quad (1.012)$$

where r takes all integral values such that $(2r+1)$ is not greater than both $R(\mu+1)$ and unity; and $(2r+1) > -2m+1$, where $(-2m)$ is negative and $< R(\mu)$.

The remainder term in (1.012), *viz.*

$$\frac{1}{2\pi i}\int_{-2m+1-\beta-i\infty}^{-2m+1-\beta+i\infty} \frac{\zeta(1-s)\,\zeta(s-\mu)\,a^{-s}\,ds}{2\cos \pi s/2}.$$

can be transformed into

$$\frac{a^{-1-\mu}}{2\pi i}\int_{2m+\beta-i\infty}^{2m+\beta+i\infty} \left[\frac{\zeta(s)\,\zeta(s-\mu)\cos \pi s/2}{\sin \pi(s-\mu)/2}\right]\Gamma(s)\left(\frac{a}{2\pi}\right)^{s} ds.$$

As $R(2m+\beta-1)>0$ and $R(2m+\beta-\mu-1)>0$, it follows that

$$\left[\frac{\zeta(s)\,\zeta(s-\mu)\cos \pi s/2}{\sin \pi(s-\mu)/2}\right]$$

has an upper bound M.

So the remainder term

$$\leqslant \frac{Ma^{-1-\mu}}{2\pi i}\int_{2m+\beta-i\infty}^{2m+\beta+i\infty} \Gamma(s)\left(\frac{a}{2\pi}\right)^{s} ds$$

$$\leqslant \frac{Me^{-2\pi/a}}{a^{1+\mu}}.$$

It is evident that, the smaller the value of a, the closer is the approximation for the same value of m.

As the series in (1.012) is meromorphic with respect to μ, it follows that when μ is an integer the limit of the series is to be taken.

When μ is an odd integer, the series tends to vanish and it is possible in such cases to get a functional relationship.

From (1.011), it follows that, if $\mu = 2m + 1$,

$$\sum_1^{\infty} \frac{r^{2m+1}}{e^{2\pi ar} - 1} = \frac{(-)^{m+1} a^{-2m-3}}{2\pi i} \int_{2m+2-k-i\infty}^{2m+2-k+i\infty} \frac{\zeta(1-s)\,\zeta(s-1-2m)\,a^s\,ds}{2\cos \pi s/2}.$$

From this and (1.01), it follows that

$$\sum_1^{\infty} \frac{r^{2m+1}}{e^{2\pi r/a} - 1} + (-)^m a^{2m+2} \sum_1^{\infty} \frac{r^{2m+1}}{e^{2\pi ar} - 1}$$

$$= \frac{1}{2\pi i}\left[\int_{k-i\infty}^{k+i\infty} - \int_{2m+2-k-i\infty}^{2m+2-k+i\infty}\right]\left[\frac{\zeta(1-s)\,\zeta(s-1-2m)\,a^s\,ds}{2\cos \pi s/2}\right]. \qquad (1.013)$$

The difference of integrals in (1.013) can always be evaluated in finite terms. The form of these terms varies with the value of m.

The poles between the two lines are all simple unless $m = -1$.

If $m > 0$, the only poles are

$$s = 2m + 2,\ 0.$$

Hence the difference of integrals

$$= + \frac{\zeta(-2m-1)\,a^{2m+2}}{2\cos \pi(m+1)} - \frac{\zeta(-1-2m)}{2}$$

$$= \frac{\Gamma(2m+2)\,\zeta(2m+2)}{(2\pi)^{2m+2}}\,\{a^{2m+2} + (-)^m\}. \qquad (1.014)$$

If $m = 0$, the poles are

$$s = 2,\ 1,\ 0.$$

Then the value of right-side of (1.013) is

$$\frac{\zeta(-1)\,a^2}{-2} - \frac{\zeta(0)\ \zeta(0)}{\pi \sin \pi/2} - \frac{\zeta(-1)}{2}$$

$$= \frac{\Gamma(2)\,\zeta(2)}{4\pi^2}(a^2 + 1) - \frac{[\zeta(0)]^2\,a}{\pi}$$

$$= \frac{a^2+1}{24} - \frac{a}{4\pi}. \qquad (1.015)$$

If $m = -1$, the simple poles are $s = \pm 1$, and the double pole is $s = 0$.

So the value in question

$$= -\tfrac{1}{2}\log a - \frac{\zeta(0)\,\zeta(2)}{2\pi}\left(a - \frac{1}{a}\right)$$

$$= -\tfrac{1}{2}\log a + \frac{\pi}{24}\left(a - \frac{1}{a}\right). \tag{1.016}$$

If $m < -1$, the poles are $2m+2,\ 0,\ 1,\ -1,\ -3,\ \ldots (2m+1)$.

Hence the value of the difference of the integrals

$$= \frac{\zeta(-1-2m)}{2}\left\{(-)^{m+1}a^{2m+2} - 1\right\}$$

$$+ \sum_{r=0}^{r=-m} \frac{a^{-2r+1}\,\zeta(2r)\,\zeta(-2m-2r)}{-\pi\sin(-2r+1)\pi/2} \tag{1.017}$$

Similarly from (1.10), it follows that

$$\sum \frac{(-)^{r-1}\, r^{\mu}}{e^{\pi a r} - e^{-\pi a r}} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\zeta(1-s)\,.\,(1-2^{-s})\,(a/2)^{-s}}{2\cos\pi s/2}$$

$$\times\ \left[\Sigma\,(-)^{r-1}\, r^{\mu-s}\right] ds.$$

The integrand and integral are convergent, if

$$R(k-\mu) > 0,\quad R(k-1) > 0.$$

The value of the integral is

$$\frac{1}{2\pi i}\int_{(L)} \frac{\zeta(1-s)\,\zeta(s-\mu)\,(1-2^{-s})\,(1-2^{1-s+\mu})\,(a/2)^{-s}\,ds}{2\cos\pi s/2}$$

$$= -\frac{2^{\mu+1}}{2\pi i}\int_{(L)} \frac{\zeta(1-s)\,\zeta(s-\mu)\,(1-2^{-s})(1-2^{s-\mu-1})\,.\,a^{-s}\,ds}{2\cos\pi s/2} \tag{1.101}$$

$$= -\frac{2^{\mu+1}}{2\pi i}\int_{\mu+1-k-i\infty}^{\mu+1-k+i\infty} \frac{\zeta(s-\mu)(\zeta(1-s)\ (1-2^{s-\mu-1})\,(1-2^{-s})\,a^{s-\mu-1}\,ds}{2\cos\pi(s-\mu-1)/2}. \tag{1.102}$$

If μ is not integral, it is possible to obtain an asymptotic series for small values of a.

$$\sum_{1}^{\infty} \frac{(-)^{r-1}\, r^{\mu}}{e^{\pi a r} - e^{-\pi a r}}$$

$$= \sum \frac{(a/2)^{-2r-1}\, \zeta(-2r)\, \zeta(2r+1-\mu)\, (1-2^{-2r-1})(1-2^{-2r+\mu})}{-\pi \sin \overline{2r+1}\,.\,\pi/2}$$

$$= \frac{2^{\mu+1}}{2\pi i} \int_{-2m+1-\beta-i\infty}^{-2m+1-\beta+i\infty} \frac{\zeta(1-s)\, \zeta(s-\mu)\, (1-2^{-s})\, (1-2^{s-\mu-1})\, a^{-s}\, ds}{2 \cos \pi s/2}, \quad (1.103)$$

where r takes all integral values such that $(2r+1) <$ the greater of $R(\mu+1)$ and unity; and $2r+1 > -2m+1$, where $(-2m+1) <$ smaller of $R(\mu)$ and unity.

When μ is an odd integer,

$$\sum \frac{(-)^{r-1}\, r^{2m+1}}{e^{\pi a r} - e^{-\pi a r}}$$

$$= \frac{(-)^{m}\, 2^{2m+2}}{2\pi i} \int_{2m+2-k-i\infty}^{2m+2-k+i\infty} \frac{\zeta(s-2m-1)\, \zeta(1-s)\, (1-2^{s-2m-2})\, (1-2^{-s})\, a^{s-2m-2}\, ds}{2 \cos \pi s/2}.$$

From this and (1.101), it follows that

$$\sum \frac{(-)^{r-1}\, r^{2m+1}}{e^{\pi r/a} - e^{-\pi r/a}} + (-)^{m}\, a^{2m+2} \sum \frac{(-)^{r-1}\, r^{2m+1}}{e^{\pi r a} - e^{-\pi r a}}$$

$$= -\frac{2^{m+2}}{2\pi i} \left[\int_{k-i\infty}^{k+i\infty} - \int_{2m+2-k-i\infty}^{2m+2-k+i\infty} \right] \times$$

$$\left[\frac{\zeta(1-s)\, \zeta(s-2m-1)\, (1-2^{-s})\, (1-2^{s-2m-2})\, a^{s}\, ds}{2 \cos \pi s/2} \right] \quad (1.104)$$

This last value can always be integrated in finite terms.

In general it is equal to

$$\sum_{r=0}^{-m} \frac{a^{-2r+1}\, \zeta(2r)\, \zeta(-2m-2r)\, (1-2^{2r-1})\, (1-2^{-2r-2m-1})\, 2^{2m+2}}{\pi \sin \pi(-2r+1)/2}$$

$$= \sum_{r=0}^{-m} \frac{(-)^{r}\, a^{-2r+1}}{\pi}\, \zeta(2r)\, \zeta(-2m-2r)\, (1-2^{2r-1})\, (2^{2m+2} - 2^{1-2r})$$

r taking integral values so that $(-2r+1) \leqslant$ the greater of $2m+1$ and unity; it will be seen that the number of terms which do not vanish is $(1-m)$.

If $m \geqslant 1$, then the value of the series in (1.104) is *nil*.

Utilizing the result of (1.20) it can be deduced that

$$\sum_{r=0}^{\infty} \frac{(2r+1)^{\mu}}{e^{\pi a(2r+1)}+1} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\zeta(1-s)\zeta(s-\mu)(1-2^{1-s})(1-2^{\mu-s})(a/2)^{-s}\,ds}{2\cos \pi s/2} \tag{1.201}$$

if $\qquad R(k-\mu-1)>0$ and $R(k)>0$;

it is also equal to

$$\frac{-2^{\mu}}{2\pi i}\int_{\mu+1-k-i\infty}^{\mu+1-k+i\infty} \frac{\zeta(s-\mu)\zeta(1-s)(1-2^{s-\mu})(1-2^{1-s})\,a^{s-\mu-1}\,ds}{2\cos(s-\mu-1)\,\pi/2}. \tag{1.202}$$

So the asymptotic expression for small values of a is

$$\frac{\zeta(-\mu)(1-2^{-\mu})(1-2^{-1})(a/2)^{-\mu-1}}{2\cos(\mu+1)\,\pi/2} - \frac{\zeta(-\mu)(1-2)(1-2^{\mu})}{2}$$

$$-\sum_{r} \frac{(a/2)^{-2r-1}\zeta(-2r)\zeta(2r+1-\mu)(1-2^{-2r})(1-2^{\mu-2r-1})}{\pi \sin \pi/2\,(2r+1)}$$

$$+\frac{1}{2\pi i}\int_{-2m+1-\delta-i\infty}^{-2m+1-\delta+i\infty} \frac{\zeta(1-s)\zeta(s-\mu)(1-2^{1-s})(1-2^{s-\mu})(a/2)^{-s}\,ds}{2\cos \pi s/2}. \tag{1.203}$$

The functional relation connecting $\phi(a)$ and $\phi(a^{-1})$ where $\phi(a)$ is the first member of (1.01) is for μ odd

$$\sum_{r=0}^{\infty} \frac{\overline{2r+1}^{\,2m+1}}{e^{\pi(2r+1)/a}+1} + \sum_{r=0}^{\infty} \frac{(-)^m \cdot \overline{2r+1}^{\,2m+1} \cdot a^{2m+2}}{e^{\pi a(2r+1)}+1}$$

$$= -\frac{2^{2m+1}}{2\pi i}\left\{\int_{k-i\infty}^{k+i\infty} - \int_{2m+2-k-i\infty}^{2m+2-k+i\infty}\right\}$$

$$\left[\frac{\zeta(1-s)\zeta(s-2m-1)(1-2^{1-s})(1-2^{s-2m-1})\,a^{s}\,ds}{2\cos \pi s/2}\right] \tag{1.204}$$

which, in general, is equal to

$$\frac{\zeta(-2m-1)(2^{2m+1}-1)}{2}\left\{1+(-)^m a^{2m+2}\right\}$$

$$+\sum_{r=0}^{-m}\frac{(-)_r a^{-2r+1}\zeta(2r)\zeta(-2m-2r)(1-2^{2r})(2-2^{-2r})}{\pi}$$

containing on the whole $2+(1-m)$ terms.

The equation (1.30) leads to the equation

$$\sum_{r=0}^{\infty}\frac{(-)^r(2r+1)^{\mu}}{e^{\frac{\pi a}{2}(2r+1)}+e^{-\frac{\pi a}{2}(2r+1)}}=\frac{1}{2\pi i}\int_{(L)}\frac{\eta(1-s)\,\eta(s-\mu)\,a^{-s}\,ds}{\sin\pi s/2}$$

$$R(k)>0,\ R(k-\mu)>0. \tag{1.301}$$

$$=\frac{1}{2\pi i}\int_{-k+\mu+1-i\infty}^{-k+\mu+1+i\infty}\frac{\eta(s-\mu)\,\eta(1-s)\,a^{s-\mu-1}\,ds}{\sin(-s+\mu+1)\pi/2}. \tag{1.302}$$

If a is small, the asymyptotic series would be

$$\sum_r\frac{\eta(1-2r)\,\eta(2r-\mu)\,a^{-2r}}{\pi/2\cos\pi r}+\frac{1}{2\pi i}\int_{-2m-\beta-i\infty}^{-2m-\beta+i\infty}\frac{\eta(1-s)\,\eta(s-\mu).a^{-s}\,ds}{\sin s\pi/2},$$

where $2r$ takes all integral values so that $2r$ is less than the greater of $R(\mu)$ and zero; and $2r$ is greater than $-2m$, where $-2m$ is less than the smaller of $R(\mu)$ and zero.

The functional relation, when μ is an odd integer equal to $2m+1$, is

$$\sum_{r=0}^{\infty}\frac{(-)^r(2r+1)^{2m+1}}{e^{\pi(2r+1)/2a}+e^{\pi(2r+1)/2a}}+\sum_{r=0}^{\infty}\frac{(-)^{m+r+1}(2r+1)^{2m+1}a^{2m+2}}{e^{\pi a(2r+1)/2}+e^{-\pi a(2r+1)/2}}$$

$$=\frac{1}{2\pi i}\left\{\int_{k-i\infty}^{k+i\infty}-\int_{-k+2m+2-i\infty}^{-k+2m+2+i\infty}\right\}\left[\frac{\eta(1-s)\,\eta(s-\mu)\,a^{s}\,ds}{\sin\pi s/2}\right] \tag{1.304}$$

$$=\sum\frac{\eta(1+2r)\,\eta(-2r-2m-1)\,a^{-2r}}{\pi/2\cos\pi r}, \tag{1.305}$$

where r is an integer so that it is less than the greater of $(2m+1)$ and zero; this contains $(2-m)$ non-vanishing terms. Hence if $m\geqslant 2$, the series (1.304) is zero.

Cyclic Equations.*

By M. BHIMASENA RAO.

Introduction.—This paper deals wih cyclic equations whose roots are x, $\phi_1(x)$, $\phi_2(x)$, ... $\phi_{n-1}(x)$, for the values 3, 4, 5 of n, where $\phi_n(x) = x$, and $\phi_n(x)$ denotes the nth power of the quadratic substitution $\phi(x) \equiv x^2 - a$, to which form the general quadratic expression $Ax^2 + 2Bx + C$ can be reduced.† It may be interesting to note that the late Mr. S. Ramanujan, F.R.S., gave an incomplete discussion of the problem for $n = 3$, and was aware of the solution for $n = 4$‡.

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* Read at the Twelfth Session of the Indian Science Congress, Benares, 1925.

† Let $\phi(x) = Ax^2 + 2Bx + C = y$, $\phi_2(x) = Ay^2 + By + C = z$, and so on. Writing X, Y, Z respectively for $Ax + B$, $Ay + B$, $Az + B$, and a for $B^2 - AC - B$, we have $X^2 = a + Y$, $Y^2 = a + Z$, etc., [*vide* Cajori's *Theory of Equations*, § 176].

‡ The case of $n = 3$ is discussed in Art 176 of Cajori's *Theory of Equations* published in 1904, which Mr. Ramanujan had not seen in all probability. In the Journal of the Indian Mathematical Society, Vol. VI, 1914, page 74, Mr. Ramanujan showed that if $x^2 = a + y$, $y^2 = a + z$, $z^2 = a + x$, then x, y, z are the roots of the cubic equation.

$$2x^3 + x^2(1 - \sqrt{4a-7}) - x(2a + 1 + \sqrt{4a-7}) + a - 2 + a\sqrt{4a-7} = 0,$$

or $$2x^3 + x^2(1 + \sqrt{4a-7}) - x(2a + 1 - \sqrt{4a-7}) + a - 2 - a\sqrt{4a-7} = 0.$$

He did not give, however, the general solution of the above equations, and had to establish by actual verification the truth of the following examples which he had proposed earlier for solution :—

$$1 + 2\sqrt{3}\,.\sin 20^\circ = \sqrt{8 - \sqrt{8 + \sqrt{8 - (1 + 2\sqrt{3}\,.\sin 20^\circ)}}}.$$

$$1 + 4\sin 10^\circ = \sqrt{11 - 2\sqrt{(11 + 2\sqrt{11 - 2(1 + 4\sin 10^\circ)})}}.$$

$$1 + 4\sqrt{3}\,.\sin 20^\circ = \sqrt{23 - 2\sqrt{(23 + 2\sqrt{(23 + 2(1 + 4\sqrt{3}\sin 20^\circ))})}}.$$

In the same Journal for 1915, Vol. VII, page 240, he stated that

$$x = \sqrt{[5 + \sqrt{\{5 + \sqrt{(5 - \sqrt{5 + x})}\}}]},$$

then $$x = \tfrac{1}{2}(2 + \sqrt{5} + \sqrt{15 - 6\sqrt{5}});$$

and if $$x = \sqrt{[5 + \sqrt{\{5 - \sqrt{(5 - \sqrt{5 + x})}\}}]},$$

then $$x = \tfrac{1}{4}\{\sqrt{5} - 2 + \sqrt{13 - 4\sqrt{5}} + \sqrt{50 + 12\sqrt{5} - 2\sqrt{65 - 20\sqrt{5}}}\}.$$

These examples obviously relate to the case $n = 4$.

Soon after, Mr. M. T. Naraniengar, simplified Mr. Ramanujan's analysis by applying an interesting result which he enunciated thus* :—

If $f(\xi)$ is a cyclic equation of the nth degree whose roots are x, $\phi(x)$, $\phi_2(x)$ ... $\phi_{n-1}(x)$, where $\phi(x)$ is of degree p, then

$$f\{\phi(x)\} = (-1)^{n(p-1)} f(x_1) \cdot f(x_2) \dots f(x_p),$$

where $x_1, x_2, x_3 \dots x_p$ are the p *values of* ξ *in the equation*

$$\phi(\xi) - \phi(x) = 0.$$

I have found that the application of this result saves much labour in the construction of the cyclic expression $f(\xi)$ of the fourth and fifth degrees.

2. Let us denote $\phi(x)$, $\phi_2(x)$ by y, z, and consider the cyclic relations of the third order.

$$x^2 = a + y \quad \dots \quad \dots \quad \dots \quad (1)$$

$$y^2 = a + z \quad \dots \quad \dots \quad \dots \quad (2)$$

$$z^2 = a + x. \quad \dots \quad \dots \quad \dots \quad (3)$$

Here x, y, z are unequal, otherwise the order of the relations would be reduced.

We write $\qquad x + y + z = t, \quad \dots \quad \dots \quad (4)$

and proceed to eliminate x, y, z from (1), (2), (3), (4).

Denoting by s_n and p_n, the sum of the nth powers, and the sum of the products taken n at a time, of the quantities x, y, z, and adding (1), (2), (3), we have,

$$s_2 = 3a + t. \quad \dots \quad \dots \quad \dots \quad (5)$$

Multiplying (1), (2), (3) by x, y, z respectively, and adding, we get,

$$s_3 = at + p_2. \quad \dots \quad \dots \quad \dots \quad (6)$$

By taking the sum of the squares of (1), (2), (3), we find

$$s_4 = 3a^2 + 2at + s_2. \quad \dots \quad \dots \quad (7)$$

Hence by Newton's formulæ,

$$2p_2 = t^2 - s_2 = t^2 - t - 3a; \quad \dots \quad \dots \quad (8)$$

$$6p_3 = t^3 - 3t \cdot s_2 + 2s_3$$

$$= t^3 - 2t^2 - 7at - t - 3a; \quad \dots \quad \dots \quad (9)$$

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* *Some Properties of Cyclic Equations:* J. I. M. S., Vol. VII, 1915, pages 7–13.

Vide: *A Special Class of Equations:* J.I.M S., Vol. VII, 1915, pages 82—88.

and $$24\,p_4 = t^4 - 6t^2 s_2 + 8t s_3 + 3s_2^2 - 6s_4$$
$$= t^4 - 2t^3 - t^2(10a+1) - 6t(a+1) - 9a(2-a). \quad (10)$$

But p_4 vanishes, therefore (10) is the eliminant and is of the fourth degree in t. This degree is too high, and the correct degree can be determined *a priori*. The degree of the eliminant is $(2^n - 2)/n$ when n is prime; it is complicated when n is composite.*

Looking for factors of (10), we find that the expression may be written

$$(t^2 - 3t - 9a)\ (t^2 + t + 2 - a). \quad \ldots (11)$$

The extraneous factor in (11) is $(t^2 - 3t - 9a)$, being a numerical multiple of the semi-invariant H of the cyclic equation

$$\xi^3 - t\,\xi^2 + p_2\,\xi - p_3 = 0, \quad \ldots \quad \ldots (12)$$

whose roots are x, y, z. The eliminant is therefore

$$t^2 + t + 2 = a. \quad \ldots \quad \ldots (13)$$

If however we take the product of the differences of (1), (2), (3), we have

$$(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = (y-z)(z-x)(x-y)$$

giving the interesting relation

$$(x+y)(y+z)(z+x) = 1. \quad \ldots \quad \ldots (14)$$

From the identity,

$$(x+y+z)^3 = x^3 + y^3 + z^3 + 3(x+y)(y+z)(z+x),$$

we get $$t^3 = s_3 + 3 = at + p_2 + 3, \text{ from (6)} \quad \ldots (15)$$

and on substituting the value of p_2 given by (8), the eliminant presents itself as

$$2t^3 - t^2 + t(1-2a) - 3(2-a) = 0,$$

or, $$(t^2 + t + 2 - a)(2t - 3) = 0, \quad \ldots \quad \ldots (16)$$

with the extraneous factor $(2t - 3)$.

3. In order to obtain the eliminant free from extraneous factors, Mr. S. Ramanujan and also Mr. Cajori first determine the sextic

$$\frac{\phi_3(x) - x}{\phi(x) - x} \equiv \frac{\{(x^2-a)^2 - a\}^2 - a - x}{x^2 - a - x} \quad \ldots (17)$$

* *Vide*: Dickson's *Theory of Numbers*, Vol. 1, page 84.

and utilise the fact that this sextic is resolvable into two cubic cyclic expressions of the form given in (12). This process is found to be tedious for high values of n, the degree of the numerator in (17) being 2^n. The superiority of Mr. M. T. Naraniengar's method consists in giving the eliminant without any reference to the sextic of (17).

From the result given at the end of the introduction to this paper, we easily see that

$$f(\xi^2 - a) = -f(\xi) \cdot f(-\xi) \quad \dots \quad \dots (18)$$

where $f(\xi)$ stands for the expression in (12).

If λ, μ are the roots of the equation

$$\xi^2 - \xi - a = 0,$$

so that

$$\left.\begin{aligned} \lambda + \mu &= 1, \\ \lambda\mu &= -a \end{aligned}\right\} \quad \dots \quad \dots (19)$$

and

then $\quad -f(-\lambda) = -f(-\mu) = 1 \quad \dots \quad \dots (20)$

that is $\quad \lambda^3 + t\lambda^2 + p_2\lambda + p_3 = 1 \quad \dots \quad \dots (21)$

and $\quad \mu^3 + t\mu^2 + p_2\mu + p_3 = 1. \quad \dots \quad \dots (22)$

Dividing the difference of (21) and (22) by $\lambda - \mu$, and using (19) we get

$$-t - p_2 = a + 1. \quad \dots \quad \dots (23)$$

If in this, we substitute the value of p_2 given by (8), we have at once the eliminant

$$t^2 + t + 2 = a \ \dots (13)$$

Now, p_3 can be found directly from (21) and (22), without using Newton's formula. Eliminating p_2 from (21) and (22) and proceeding as before we obtain

$$-at - p_3 = a - 1. \quad \dots \quad (24)$$

From (13), we see that a is a rational and integral function of t. On substituting the value of a in (23) and (24), p_2 and p_3 are expressed in terms of t and the cyclic equation given in (12) is therefore expressed as

$$f(\xi) \equiv \xi^3 - t\xi^2 - (t^2 + 2t + 3)\xi + (t^3 + 2t^2 + 3t + 1) = 0. \quad (25)$$

4. To solve equation (25), we proceed in the usual way by expressing $f(\xi)$ in the standard form $(a_0, a_1, a_2, a_3)\ (\xi, 1)^3$,

where $\quad a_0 = 1,\ a_1 = -\frac{1}{3}t,$

$$a_2 = -\tfrac{1}{3}(t^2 + 2t + 3),$$

and $\quad a_3 = t^3 + 2t^2 + 3t + 1,$

whence $H \equiv a_0a_2 - a_1^2 = -\frac{1}{9}(4t^2 + 6t + 9)$, ... (26)

$$G \equiv a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3 = \frac{1}{27}(16t^3 + 36t^2 + 54t + 27),$$

$$= \frac{1}{27}(4t^2 + 6t + 9)(4t + 3), \quad \text{... (27)}$$

$$= -HK$$

where K stands for $\frac{1}{3}(4t + 3)$. ... ... (28)

From (26) and (28), we have,

$$H = -\frac{1}{4}(K^2 + 3) \quad \text{... ... (29)}$$

Also the discriminant $D \equiv \frac{1}{a_0^2}(G^2 + 4H^3)$

$$= -3H^2, \text{ from (27) and (29)} \quad \text{... (30)}$$

By taking $\tan 3\theta = \frac{K}{\sqrt{3}}$, we easily find that the roots of equation (25) are,

$$\frac{t}{3} + \frac{\sqrt{3}.\sin\theta}{\cos 3\theta};\ \frac{t}{3} + \frac{\sqrt{3}.\sin(2\pi/3+\theta)}{\cos 3\theta};\ \frac{t}{3} + \frac{\sqrt{3}.\sin(4\pi/3+\theta)}{\cos 3\theta}. \quad (31)$$

where $\theta = \frac{1}{3}\tan^{-1}\left(\frac{4t + 3}{3\sqrt{3}}\right)$,

and $t^2 + t + 2 = a$.

We may take the first value in (31) to be x, and it is only by actual calculation that we can determine which of the remaining two represents y or z.* In fact, the third root in (31) turns out to be y, and we therefore write

$$\left.\begin{aligned} x &= \frac{t}{3} + \frac{\sqrt{3}\sin\theta}{\cos 3\theta} \\ y &= \frac{t}{3} + \frac{\sqrt{3}.\sin(4\pi/3 + \theta)}{\cos 3\theta} \\ z &= \frac{t}{3} + \frac{\sqrt{3}.\sin(8\pi/3 + \theta)}{\cos 3\theta} \end{aligned}\right\} \quad \text{... ... (32)}$$

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* From (31), $\frac{t}{3} = \frac{1}{4}(\sqrt{3}\tan 3\theta - 1)$.

$$\therefore \quad x = \frac{\sqrt{3}\tan 3\theta - 1}{4} + \frac{\sqrt{3}\sin\theta}{\cos 3\theta},$$

It is to be noticed that a change from x to y, y to z, or z to x increases θ by $\frac{4\pi}{3}$.

5. The cyclic relations (1), (2) and (3) can be written in the radical form

$$z = \pm\sqrt{a+x},$$

$$y = \pm\sqrt{a \pm \sqrt{a+x}},$$

$$x = \pm\sqrt{a \pm \sqrt{[a \pm \sqrt{(a+x)}]}}.$$

$$\therefore \quad x^2 = \frac{3\tan^2 3\theta + 1 - 2\sqrt{3}.\tan 3\theta}{16} + \frac{3\sin^2\theta}{\cos^2 3\theta} + \frac{\sqrt{3}.\sin\theta}{2\cos 3\theta}(\sqrt{3}\tan 3\theta - 1)$$

$$a = \frac{(3\sqrt{3}\tan 3\theta - 3)^2}{16} + \frac{3\sqrt{3}\tan 3\theta - 3}{4} + 2.$$

$$\therefore \quad y = x^2 - a$$

$$= -\frac{(\sqrt{3}\tan 3\theta - 1)^2}{2} - 3.\frac{(\sqrt{3}\tan 3\theta - 1)}{4} - 2 + 3\frac{\sin^2\theta}{\cos^2 3\theta} + \frac{\sqrt{3}\sin\theta(\sqrt{3}\tan 3\theta - 1)}{2\cos 3\theta}$$

$$= \frac{\sqrt{3}\tan 3\theta - 1}{4} - \frac{(\sqrt{3}\tan 3\theta - 1)^2}{2} - (\sqrt{3}\tan 3\theta - 1) - \tfrac{1}{2} - \tfrac{3}{2} + \frac{3\sin^2\theta}{\cos^2 3\theta} + \frac{\sqrt{3}\sin\theta.(\sqrt{3}\tan 3\theta - 1)}{2\cos 3\theta}$$

$$= \frac{t}{3} - \frac{3\tan^2 3\theta + 3}{2} + \frac{3\sin^2\theta}{\cos^2 3\theta} + \frac{\sqrt{3}\sin\theta.(\sqrt{3}\tan 3\theta - 1)}{2\cos 3\theta}$$

$$= \frac{t}{3} + \frac{-3 + 6\sin^2\theta + 3\sin\theta.\sin 3\theta - \sqrt{3}\sin\theta\cos 3\theta}{2\cos^2 3\theta}$$

$$= \frac{t}{3} + \frac{-3 + 3(1 - \cos 2\theta) + 3\sin\theta\sin 3\theta - \sqrt{3}\sin\theta\cos 3\theta}{2\cos^2 3\theta}$$

$$= \frac{t}{3} - \frac{3\cos\theta\cos 3\theta + \sqrt{3}.\sin\theta\cos 3\theta}{2\cos^2 3\theta}$$

$$= \frac{t}{3} - \frac{3\cos\theta + \sqrt{3}\sin\theta}{2\cos 3\theta}$$

$$= \frac{t}{3} + \frac{\sqrt{3}.\sin(\theta + 4\pi/3)}{\cos 3\theta}.$$

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NOTES AND QUESTIONS.

Notes and Questions.

On a Query of Dr. Muirhead.

1. Dr. Muirhead has determined* the number $\phi_r(n)$ of convex regions into which S_r (a flat space of r dimensions) is divided by any n of S_{r-1}'s in general position in it. Denoting by $\chi_r(n)$, the number of completely closed non-overlapping regions whch are formed in a S_r by n S_{r-1}'s in general position, the following formulæ are given by him with a query:—

$$\chi_r(n) = {}_{n-1}C_r \tag{1.1}$$

$$\phi_r(n) = \phi_{r-1}(n) + {}_nC_r \tag{1.2}$$

$$\therefore \quad \phi_r(n) = \phi_{r-1}(n) + \chi_r(n+1). \tag{1.3}$$

Query:—*Can this be interpreted directly* ?

It is the object of this note to obtain such an interpretation of the formula (1.3).

2. Definitions—

A *prime* in S_r is a flat space of $(r - 1)$ dimensions.

A *half-line* or *ray* is a line proceeding to infinity one way only and terminated the other way. It has thus but one direction. A point at a very great distance from the terminal point we shall call an *infinite point*† of the ray. A line infinite both ways is to be thought of as two rays in juxta-position and has thus two opposite directions, and two *opposite infinite points*. Two such points are on opposite sides of any prime, since every prime divides any line not on it into two half-lines. (2.1)

A *region* in S_r with n primes is the totality of points such that the finite line segment joining any two of them has no point in common with any of the given n primes. (2.2)

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* Note on Topics of regions separated by Boundaries: *Proceeding of the Edinburgh Math. Society*, Vol. XLIII, Part II, pp 111—114.

† The word "infinite" is used throughout this paper in the sense "sufficiently far off from the finite regions of S_r to serve the purpose in view."

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A region is *closed* if every line in the region is of finite length ; otherwise, it is *open*. An open region will, therefore, contain one or more half-lines. (2.3)

Two open regions are *opposite* if they contain opposite infinite points.

3. The following results which are easily seen to be true will be found useful. It is assumed that, all the primes considered are in general position in S_r; that is, no two of them are parallel, no three have a common S_{r-1} and so on

(i) *If two points belong to the same region they lie on the same side of each of the given n primes.* (3.1)

(ii) *Two points belonging to different regions must lie on opposite sides of at least one of the given primes.* (3.2)

Both these follow readily from the definition of a region given in (2.2)

(iii) $(r+1)$ *primes in general position determine a closed region and conversely, every closed region in S_r is bounded by at least $(r+1)$ primes.* (3.3)

(iv) *Every open region has a unique opposite open region.* (3.4)

If possible let the region R_1 be opposite to both R_2 and R_3. Then there exist two lines P_1P_2 and Q_1Q_3 whose infinite points P_1Q_1 lie in R_1 while the opposite infinite points P_2 and Q_3 lie in R_2 and R_3 respectively. Since R_2 and R_3 are distinct regions there exists a prime p which separates Q_2 and Q_3. Now P_1 and P_2 are on opposite sides of p being opposite infinite points (2.1), P_2 and Q_3 are on opposite sides of p by hypothesis, while Q_3 and Q_1 are also on opposite sides of p being opposite infinite points. Hence it would follow that P_1 and Q_1 should be on opposite sides of p which cannot be since they belong to the same region R_1 (2.2)

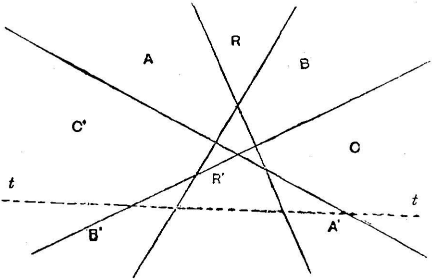
(v) *Any prime which does not meet an open region R and which is far away from the finite regions of S_r meets the opposite region R′ and closes it.* (3.5)

For any such prime t is met by *every* line in one or other of its two opposite infinite points. Since t does not meet R it cannot contain any of the infinite points in R, and must therefore contain all the infinite points in the opposite region R′. Thus t meets every ray in R′ and it follows that if t be adjoined to the existing bounding primes of R′ the region becomes closed since all lines in it are now of finite length.

4. We are now in a position to perceive *directly* the truth of Dr. Muirhead's result

Consider n primes in S_r. These form $\phi_r(n)$ regions some of which may be closed and some open. Take any transversal prime t so far away that it does not meet any of the closed regions, (if such exist). Of the open regions it meets some but not necessarily all. Let us reckon separately the number of regions which t meets and the number which it does not meet. The sum of these numbers must obviously be $\phi_r(n)$.

Every open region which t meets cuts out a corresponding region in it, so that the number of regions which it meets is also the number of region determined in t by the n S_{r-2}'s in which it is cut by the given n primes. This is by definition $\phi_{r-1}(n)$.



The region t fails to meet consist of (*a*) all the closed regions and (*b*) some of the open regions like A, B, R. (see figure). But from (3.5) for every open region R which t does not meet there is an opposite open region R′ which it meets and closes, so that the number in (*b*) is the number of *additional* closed regions obtained by the adjunction of t to the existing n primes. Hence the total number of regions not met by t is the number of closed non-overlapping regions in a S_n with $(n+1)$ primes, namely, the existing n primes and t. Hence this number is $\chi_r(n+1)$.

$$\therefore \quad \phi_r(n) = \phi_{r-1}(n) + \chi_r(n+1).$$

The above proof applies for all values of n and r.

5. We have seen (3.5) that whenever a distant prime t fails to meet an open region R, it meets its opposite R′ and closes it. It is easy to see

conversely, that when t closes a region R′, it cannot meet its opposite R since it cannot, in general, contain opposite infinite points. When such a closure occurs, t forms one of the boundaries of the newly-closed region, and hence a closed region is cut out on t by R′. Hence we have also the following results :—

The number of open regions in a S_r *with n primes is* equal to the number of open regions met by t plus the number of open regions not met by it, and is hence the *total number of regions formed by n* S_{r-2}'s *in a* S_{r-1} *each closed region being counted twice.*

6. Again, since for every additional closed region obtained by the adjunction of t to the existing primes, we have a corresponding closed region determined on t and *vice versa*, it follows that

$$\chi_r(n+1) - \chi_r(n) = \chi_{r-1}(n)$$

or changing n into $n-1$

$$\chi_r(n) = \chi_r(n-1) + \chi_{r-1}(n-1) \qquad (5.1)$$

The relation (5.1) is of interest enabling us to express $\chi_r(n)$ as an explicit function of n and r. In fact if we construct a tabular form in which $\chi_r(n)$ is the element in the nth row and the rth column, it appears from (5.1) that each element of the array is the sum of two elements in the preceding row, *viz.*, the one above it and the one on its left. It is easy to see that with such a law of formation the several elements are uniquely determined in terms of those in the first row and those in the first column and that these may be chosen at pleasure, being in fact the arbitrary constants of a general solution of the functional equation (5.1). In particular it follows that if two solutions of (5.1) are equal all along the first row and the first column, they must be identical.

Now $\chi_r(1)$ = the number of closed regions formed by one prime in S_r
= 0 for all values of r

and $\chi_1(n)$ = the number of finite segments into which a straight line is divided by n points on it
$= n-1$

Also as is well-known, the binomial co-efficients ${}_{n+k}C_r$ satisfy (5.1) for all values of k and for the particular value $k = -1$ we have in addition

$${}_{n-1}C_r = n - 1 = \chi_r(n)$$

for $r = 1$ that is down the first column and

$${}_{n-1}C_r = 0 = \chi_r(n)$$

for $n = 1$ that is all along the first row.

Hence $\chi_r(n) \equiv {}_{n-1}C_r$ (1·1)

and
$$\begin{aligned}\phi_r(n) &= \phi_{r-1}(n) + \chi_r(n+1)\\ &= \phi_{r-1}(n) + n\,C_r\\ &= 1 + n\,C_1 + nC_2 + \dots nC_r\end{aligned}$$

since $\phi_1(n) = n + 1.$

Both the results are obtained by Dr. Muirhead by the method of induction.

V. THIRUVENKATAÇHARYA
and
A. NARASINGA RAO.

A New Rule for the Expansion of a Determinant:

1. In the determinant of n rows and n columns, n letters are chosen so that no two are in the same row or column. The product of these n letters is taken with a sign given according to a prescribed rule. The object of this note is to give a new and simpler rule for finding the sign of the partial product.

RULE I. Mark the n letters, no two of which are in the same row or column. From the marked letter in the first column proceed to the marked letters in the succeeding columns, noting the number of ascents* in each case and *ignoring descents*. From the marked letter in the second column proceed to the marked letters in the succeeding columns, noting similarly the number of ascents and ignoring descents. In the same manner deal with the marked letters in the third and succeeding columns. Then take the total number of ascents. If this number be *zero* or an *even* number, the sign of the partial product is *positive*. If this number is *odd*, the sign of the product is *negative*.

Illustration :—Suppose we wish to find the sign of the term *enkd* in the determinant

$$\begin{vmatrix} a & b & c & \underline{d} \\ \underline{e} & f & g & h \\ i & j & \underline{k} & l \\ m & \underline{n} & o & p \end{vmatrix}$$

* The *number* of steps in an ascent is immaterial and we count only *one ascent* whatever the number of steps in an ascent.

We mark the letters e, n, k and d. We proceed from e to n. This is a descent. Then we proceed from e to k. This is also a descent. Then we proceed from e to d. This is an ascent. So in going from e to the succeeding marked letters, the total number of ascents is *one*.

Next we proceed similarly with the marked letter in the second column to marked letters in the succeeding columns. In proceeding from n to k we have one ascent and from n to d we have another. So in proceeding from n to the marked letters in the succeeding columns, we have two ascents.

Again we proceed from k to d and we get an ascent.

Hence the total number of ascents is *four*. Thus the sign of the product $e\,n\,k\,d$ is *positive*.

2. To show that 'rows' and 'columns' can be interchanged in any proposition about determinants, we may give another rule and show afterwards that the two rules are eqivalent.

RULE II. In Rule I, replace the word 'column' by 'row' and 'descent' by 'forward march from left to right' and 'ascent' by 'backward march from right to left.' This gives the second Rule which may be stated thus :—Start from the marked letter in the first row and proceed to marked letters in succeeding rows noting the number of backward marches and ignoring forward marches. Similarly, treat the marked letters in all the other rows If the total number of backward marches is *zero* or an *even* number the sign is *positive*; otherwise, the sign is *negative*.

Illustration :—Consider two term *enkd* as before in the determinant

$$\begin{vmatrix} a & b & c & \underline{d} \\ \underline{e} & f & g & h \\ i & j & \underline{k} & l \\ m & \underline{n} & o & p \end{vmatrix}$$

In proceeding from the marked letter d in the first row, to marked letters e, k, n, we have to note the number of backward marches, ignoring forward marches. Thus from d to e, we have a backward march : from d to k, we have another and from d to n, we have a third. So in going from

d to the letters in the succeeding rows, we go from right to left three times. Then take e. From e to k, we go from left to right and so we ignore it From e to n also we go from left to right. So the number of backward marches is *nil*. Then from the letter k in the third row we proceed to the letter n in the succeeding row. We go from right to left and so we take note of it. So altogether corresponding to the product *enkd*, we have four backward marches Hence the sign of *enkd* is *positive*.

3. That the above two rules are equivalent is seen easily; for, if there is an ascent from one marked letter to a marked letter in a succeeding column in Rule I, there is a backward march from a marked letter to a marked letter in a succeeding row in Rule II.

4. In most elementary text-books like Hall and Knight's *Higher Algebra*, the theoretical treatment of determinants of higher order than three is entirely omitted. This is due to the usual rule about the expansion of determinants being cumbrous. For the application of that rule, the elements of the determinant have to be written a_1, b_1, c_1, a_2, b_2, c_2., etc. But with our rule, we can deal with any determinant, the elements being expressed in any manner.

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Solutions.

Question 1252.

(K. J. SANJANA, M.A.) :—Prove that for real integral values of x and y the equality

$$6x^2 + 2 = y^3$$

is impossible, except in the single case when $x = 1$ and $y = 2$.

Examine if the equality holds for any other real rational values of x and y.

Solution by S. D. Chowla.

Since $y^3 = 6x^2 + 2$, y must be even.

Put $y = 2z$. The equation becomes

$$4z^3 = 3x^2 + 1.$$

It is obvious that x must be odd.

Put $x = 2s + 1$;

We have

$$z^3 = 3s^2 + 3s + 1$$

or

$$z^3 + s^3 = (s + 1)^3,$$

which is impossible for integral values of s and z by Fermat's Theorem, except for $s = 0$, $z = 1$; that is, $x = 1$, $y = 2$.

Question 1338.

(Selected : PROF. CAYLEY):—Show that the surfaces

$$xyz = 1,\ yz + zx + xy + x + y + z + 3 = 0$$

intersect in two distinct cubic curves; and find the equations of the cubic cones which have their vertices at the origin and pass through these curves respectively.

Solution by N. Durairajan.

The equations may be written as

$$yz = \frac{1}{x}, \quad \ldots \quad \ldots \quad \ldots (1)$$

and

$$yz + (y + z)(x + 1) + x + 3 = 0. \ldots \quad \ldots (2)$$

$$\therefore \quad y + z = -\frac{x^2 + 3x + 1}{x(x + 1)}. \quad \ldots \quad \ldots (3)$$

Solving for y and z

$$y = -\frac{x+1}{x} \text{ and } z = -\frac{1}{x+1}; \qquad \ldots \qquad \ldots (4)$$

$$y = -\frac{1}{x+1} \text{ and } z = -\frac{x+1}{x}. \qquad \ldots \qquad \ldots (5)$$

The curves (4) and (5) are cubic curves, as they cut any plane in *three* points. These two curves constitute together the complete intersection of the cubic and quadric surfaces given by (1) and (2).

The cubic cone with its vertex at the origin and passing through the curve (4) is given by eliminating x between

$$\frac{Y}{X} = -\frac{x+1}{x^2} \text{ and } \frac{Z}{X} = -\frac{1}{x(x+1)}.$$

The eliminant is

$$XY^2 + YZ^2 + ZX^2 - 3XYZ = 0. \ldots \qquad \ldots (6)$$

The equation of the other cone can be similarly obtained.

Question 1344.

(V. Tiruvenkatacharya):—Show that if $|q| < 1|$,

(i) $(1-q^4)^2(1-q^8)(1-q^{12})^2(1-q^{16}) \ldots$
$= 1 - 2q^4 + 2q^{16} - + \ldots$

(ii) $[1-q^2)(1-q^6)(1-q^{10}) \ldots]^{-1}$
$= (1+q^2)(1+q^4)(1+q^6) \ldots$

Solution by S. D. Chowla.

(i) It is well known that

$$q_0 q_3^2 = 1 - 2q + 2q^4 - 2q^9 + 2q^{16} - + \ldots$$
$$= 1 + \sum_1^\infty (-1)^n 2 . q^{n^2}$$

where $q_0 = \prod_{n=1}^\infty (1-q^{2n}),\ q_3 = \prod_{n=1}^\infty (1-q^{2n-1})$.*

Change q into q^4 and we get (i).

(ii) We have $\prod_{n=1}^\infty (1+q^n) = \dfrac{\prod_1^\infty (1-q^{2n})}{\prod_1^\infty (1-q^n)} = \dfrac{1}{\prod_1^\infty (1-q^{2n-1})}$.

Change q into q^2 and we get (ii).

* (Bromwhich: *Theory of Infinite Series*, pp. 105-106).

Question 1353.

(ENQUIRER) :—Show that

$$\int_0^\infty \frac{\sin 2nx \sin^m x \, dx}{x^{a+1}} = \frac{\pi \, \Delta^m (2n-m)^a}{2^{m+1} \Gamma(a+1) \cos \frac{1}{2}(a+m)\pi},$$

if $2n > m > a$, all being positive. (Boole)

Solution by S. D. Chowla.

A similar result is given in Carr's *Synopsis*, 2724.

We shall follow Carr's method (2717 to 2724).

1. First suppose m is even and equal to $2p$, p being an integer.

Since $$\frac{1}{x^{a+1}} = \frac{1}{\Gamma(a+1)} \int_0^\infty e^{-zx} \cdot z^a \, dz,$$

$$\int_0^\infty \frac{\sin 2nx \sin^{2p} x}{a^{a+1}} dx$$

$$= \frac{1}{\Gamma(a+1)} \int_0^\infty \int_0^\infty \sin 2nx \sin^{2p} x \, z^a e^{-zx} \, dz \, dx. \quad (1)$$

Now $$\int_0^\infty e^{-kx} \Delta^{2p} \sin(2d-2p) x \, dx = \Delta^{2p} \frac{2d-2p}{k^2+(2d-2p)^2}$$

(Carr. 2717 (i)). (2)

Also

$$\begin{aligned} \Delta^{2p} \sin(2d-2p)x &= (E-1)^{2p} \{\sin(2d-2p)x\} \\ &= \sin(2d+2p)x - {}^{2p}C_1 \sin(2d-2p-2)x \\ &\qquad + {}^{2p}C_2 \sin(2d+2p-4)x - \ldots + \sin(2d-2p)x \\ &= \sin 2dx \{\cos 2px - {}^{p}C_1 \cos(2p-2)x + \ldots + \cos 2px \\ &\quad + \cos 2dx \{\sin 2px - {}^{2p}C_1 \sin(2p-2)x + \ldots - \sin 2px\}. \end{aligned}$$

The co-efficient of $\cos 2dx$ in this expression is zero, since terms equidistant from the beginning and the end cancel each other ; the co-efficient of $\sin 2dx$ is equal to $2^{2p}(-1)^p \sin^{2p} x$, as is well-known.

∴ (2) becomes:

$$\int_0^\infty e^{-kx} \sin 2dx \sin^{2p} x \, dx = \frac{(-1)^p}{2^{2p}} \Delta^{2p} \frac{(2d-2p)}{k^2+(2d-2p)^2}$$

$$= \frac{(-1)^p}{2^{2p}} \left\{ \frac{2d+2p}{k^2+(2d+2p)^2} - \frac{{}^{2p}C_1(2d+2p-2)}{k^2+(2d+2p-2)^2} \right.$$

$$\left. + \frac{{}^{2p}C_2(2d+2p-4)}{k^2+(2d+2p-4)^2} \ldots + \frac{2d-2p}{k^2+(2d-2p)^2} \right\}. \quad (3)$$

Integrating the double integral on the right-hand side of (1), first with respect to x, and using (3), we get

$$\int_0^\infty \frac{\sin 2nx \,.\, \sin^{2p} x\, dx}{x^{a+1}} = \frac{1}{\Gamma(a+1)} \int_0^\infty \frac{(-1)^p}{2^{2p}}\, z^a\, \Delta^{2p}\, \frac{(2n-2p)}{z^2+(2n-2p)^2}\, dz$$

$$= \frac{(-1)^p}{\Gamma(a+1)\,2^{2p}} \int_0^\infty \left[\frac{(2n+2p)\,z^a}{z^2+(2n+2p)^2} - \frac{{}^{2p}C_1\,(2n+2p-2)\,z^a}{z^2+(2n+2p-2)^2} \dots + \frac{(2n-2p)\,z^a}{z^2+(2n-2p)^2}\right] dz. \qquad (4)$$

But, from the well-known result,

$$\int_0^\infty \frac{y^{p-1}}{1+y}\, dy = \frac{\pi}{\sin p\pi},$$

it follows that
$$\int_0^\infty \frac{z^{2p-1}\, dz}{a^2+z^2} = \frac{a^{2p-2}\,\pi}{2\sin p\pi}. \qquad (5)$$

Using (5), the right-hand side of (4) becomes

$$\frac{(-1)^p}{\Gamma(a+1)\,2^{2p}\,\sin(a+1)\,\pi/2}\,\{(2n+2p)^a - 2p\,C_1\,(2n+2p-2)^a + {}^{2p}C_2\,(2n+2p-4)^a + (2n-2p)^a\}. \qquad (6)$$

But the expression within the curved brackets on the right-hand side of (6) is

$$(E-1)^{2p}\,\{2n-2p\}^a = \Delta^{2p}\,(2n-2p)^a.$$

Hence
$$\int_0^\infty \frac{\sin 2nx\, \sin^{2p} x\, dx}{x^{a+1}} = \frac{(-1)^p\,\pi\,\Delta^{2p}\,(2n-2p)^a}{2^{2p+1}\,\Gamma(a+1)\cos(a+2p)\pi/2}. \qquad (7)$$

Thus the result is established when m is even.

2. Next, let m be odd and equal to $2p+1$.

Now,
$$\int_0^\infty e^{-kx}\,\Delta^{2p+1}\cos(2d-2p-1)\,x\,dx = \Delta^{2p+1}\frac{k}{k^2+(2\alpha-2p-1)^2}, \qquad (8)$$

and we get, as before,

$$\Delta^{2p+1}\cos(2\alpha-2p-1) = (-)^{p+1}\,2^{2p+1}\sin^{2p+1} x\,.\sin 2dx.$$

$$\therefore \int_0^\infty e^{-kx}\sin^{2p+1} x\,.\,\sin 2nx\, dx = \frac{(-)^{p+1}}{2^{2p+1}}\,\Delta^{2p+1}\,\frac{k}{k^2+(2n-2p-1)^2} \qquad (9)$$

$$= \frac{(-)^{p+1}}{2^{2p+1}}\left\{\frac{k}{k^2+(2n+2p+1)^2} - \frac{{}^{2p}C_1 k}{k^2+(2n+2p-1)^2} + \frac{{}^{2p}C_2 k}{k^2+(2n+2p-3)} \dots - \frac{k}{k^2+(2n-2p-1)^2}\right\}$$

Also as before,

$$\int_0^\infty \frac{\sin 2nx . \sin^{2p+1} x \, dx}{x^{a+1}},$$

$$= \frac{1}{\Gamma(a+1)} \int_0^\infty \int_0^\infty \sin 2nx \sin^{2p+1} x \, e^{-2x} . z^a \, dx \, dz.$$

$$= \frac{(-1)^{p+1}}{2^{2p+1} \Gamma(a+1)} \int_0^\infty z^a \, \Delta^{2p+1} \frac{z}{z^2 + (2n - 2p - 1)^2} \, dz$$

$$= \frac{(-1)^{p+1}}{2^{2p+1} \Gamma(a+1)} \int_0^\infty \left\{ \frac{1}{z^2 + (2n + 2p + 1)^2} \right.$$

$$\left. - \frac{{}^{2p}C_1}{z^2 + (2n + 2p - 1)^2} - \cdots \frac{-1}{z^2 + (2n - 2p - 1)^2} \right\} z^{a+} \, dz \text{ by (9)}$$

$$= \frac{(-)^{p+1} \pi}{2^{2p+2} \Gamma(a+1) \sin (a+2) \pi/2} \left\{ (2n + 2p + 1)^a \right.$$

$$\left. - {}^{2p}C_1 (2n + 2p - 1)^a - + \ldots - (2n - 2p - 1)^a \right\}$$

$$= - \frac{\pi \, \Delta^{2p+1} (2n - 2p - 1)^a}{2^{2p+2} \Gamma(a+1) \cos (a + 2p + 1) \pi/2}.$$

Thus the result is true for all positive integral values of m, except that when m is odd, the sign is wrong.

Question 1374.

(HANS R. GUPTA) :—If $\frac{p_n}{q_n}$ denote the nth convergent of the C.F.

$$\cos\theta - \frac{1}{2\cos\theta -} \frac{1}{2\cos\theta -} \frac{1}{2\cos\theta -} \cdots,$$

show that $p_n = \cos n\theta$.

Find a similar C. F. for which $p_n = \sin n\theta$.

Solution by M. Venkatarama Aiyar.

By direct simplification, we find

$$p_1 = \cos\theta,$$
$$p_2 = 2\cos^2\theta - 1 = \cos 2\theta,$$
$$p_3 = 4\cos^3\theta - 3\cos\theta = \cos 3\theta.$$

Now, assume that $p_n = \cos n\theta$

and $p_{n-1} = \cos \overline{n-1}\,\theta.$

From the law of formation of successive convergents

$$p_n = a_n p_{n-1} + b_n p_{n-2}$$

$$\therefore \quad p_{n+1} = 2\cos\theta . \cos n\theta - 1 . \cos \overline{n-1}\,\theta$$
$$= \cos \overline{n+1}\,\theta.$$

But p_1, p_2, p_3 satisfy the law.

Hence generally $$p_n = \cos n\theta.$$

Again, from the fact that

$$\sin n\theta = 2\cos\theta \sin \overline{n-1}\,\theta - \sin \overline{n-2}\,\theta$$

and that $$\sin 2\theta = 2\cos\theta\sin\theta.,$$

it is readily seen that the C. F.

$$\frac{\sin\theta}{1-}\;\frac{1}{2\cos\theta-}\;\frac{1}{2\cos\theta-}\;\cdots,$$

has its $$p_n = \sin n\theta.$$

Question 1378.

(B. B. BAGI) :—Prove that the incentre of the triangle formed by lines drawn parallel to and at distances $\frac{2}{3}r_1, \frac{2}{3}r_2$ and $\frac{2}{3}r_3$ from the sides BC, CA, AB of an acute-angled triangle ABC, coincides with the centroid of the ex-central triangle $I_1I_2I_3$; the parallel lines and the vertices being on opposite sides of BC, CA and AB.

Solution by N. P. Subramanian, M. Venkatarama Ayyar and T. S. Venkataraman.

Let $A'B'C'$ be the triangle formed by the parallel lines in the question. Also let $A''B''C''$ be the triangle formed by similar parallel lines through I_1, I_2, I_3. Then obviously the sets of points (A, A', A''), $(B, B'\,B'')$, (C, C', C'') are collinear.

Now, if S, G, O be the circum-centre, centroid and ortho-centre of $I_1I_2I_3$, O will be the in-centre of ABC. Further, since ABC is the pedal triangle of $I_1I_2I_3$, SI_1 will be perpendicular to BC.

Hence SI_1 is also perpendicular to $B''C''$.

Similarly SI_2, SI_3 are perpendicular to $C''A''$, $A''B''$.

Also $$SI_1 = SI_2 = SI_3.$$

Thus S is the in-centre of $A''B''C''$.

In other words, S and O are corresponding points in the similar triangles $A''B''C''$ and ABC.

But $SG : GO = A''A' : A'A = B''B' : B'B = C''C' : C'C.$

Hence G is the corresponding point in the triangle $A'B'C'$. That is, G is the incentre of $A'B'C'$.

(Analytical solution by Kanwar Bahadur).

Question 1381.

(K. SATYANARAYANA) :—Show that the polars of the conjugates of involution range with regard to any conic whose directrices pass through the double points form an involution pencil with the focal lines of the vertex of the pencil as double lines.

Solution by N. P. Subramaniam.

The conjugates of the involution range coincide with the involution range itself. Therefore their polars meet in a point O and since the range is in involution the pencil of polars is also in involution. The double lines of this involution pencil are the polars of the double points of the involution range. Also since the directrices pass through the double points, the polars of the double points pass through the foci of the conic. Hence the double lines of the pencil pass through the foci and are therefore the focal lines through the vertex of the pencil.

Question 1383

(MARTYN THOMAS) :—Two non-intersecting circles are inverted from one of their limiting points into two concentric circles.

If a and b be the radii of the circles and c the distance between their centres and k the radius of inversion, prove that the radii of the inverse circles will be

$$\frac{2c^2k^2a}{\Delta\,(c^2+a^2-b^2+\Delta)} \quad\text{and}\quad \frac{2c^2k^2b}{\Delta\,(c^2+b^2-a^2-\Delta)},$$

where $\Delta^2 = a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2,$

and find the position of the common centre.

Hence or otherwise show that if n circles can be described each touching the two given circles and also two of their own number,

$$c^2 = a^2+b^2 \pm 2ab\left(\tan^2\frac{\pi}{n}+\sec^2\frac{\pi}{n}\right).$$

Solution by T. S. Venkataraman and Kanwar Bahadur.

The two circles may be written

$$x^2+y^2-2g_1x+d=0,$$
$$x^2+y^2-2g_2x+d=0,$$

where $g_1-g_2=c,\ g_1^2-d=a^2,\ g_2^2-d=b^2$

$$\therefore\ g_1=\frac{c^2+a^2-b^2}{2c};\ g_2=\frac{a^2-b^2-c^2}{2c},\ d=\frac{\Delta^2}{4c^2}. \qquad (1)$$

Changing the origin to the limiting point L′ $(-\sqrt{C}, O)$, the first circle becomes

$$(x-\sqrt{d})^2+y^2-2g_1(x-\sqrt{d})+d=0.$$

Its inverse with respect to the new origin L′, is obtained by

$$\frac{k^2x}{x^2+y^2} \text{ and } \frac{k^2y}{x^2+y^2} \text{ for } x \text{ and } y,$$

and is given by

$$x^2+y^2-\frac{2k^2}{2\sqrt{d}}x+\frac{k^4}{2\sqrt{D}(g_1+\sqrt{d})}=0.$$

Its radius $$a'=\frac{k^2a}{2\sqrt{d}(g_1+\sqrt{d})}=\frac{2c^2+k^2a}{\Delta(c^2+a^2-b^2+\Delta)}, \text{ from (1)}$$

Similarly $$b'=\frac{2c^2k^2b}{\Delta(c^2+b^2-a^2-\Delta)}.$$

Next, the co-ordinates of the common centre are $(\xi, 0)$ where

$$\xi=\frac{k^2}{2\sqrt{d}}-\sqrt{d}=\frac{2c^2k^2-\Delta^2}{2c\Delta}.$$

Lastly, if a, b are the radii of two circles and c the distance between their centres, and if a', b', c' are the corresponding values of the inverse system

$$\frac{c^2-(a-b)^2}{ab}=\frac{c'^2-(a'-b'^2)^2}{a'b'},$$

(Salmon's *Conic Sections*, p. 114).

The inverse system is a pair of concentric circles touched by n equal circles.

$$\therefore \quad \frac{c^2-(a-b)^2}{ab}=\frac{0-(a'-b')^2}{a'b'},$$

$$=-4\tan^2\frac{\pi}{n}$$

since $$\frac{a'-b'}{2}=\frac{a'+b'}{2}\sin\frac{\pi}{n},$$

that is, $$b'=\frac{1-\sin\pi/n}{1+\sin\pi/n}.a'$$

$$\therefore \quad c^2=a^2+b^2-2ab-2ab\left(\tan^2\frac{\pi}{n}+\sec^2\frac{\pi}{n}-1\right)$$

$$=a^2+b^2-2ab\left(\tan^2\frac{\pi}{n}+\sec^2\frac{\pi}{n}\right).$$

Questions for Solution

1412. (K. J. SANJANA AND HARIHAR P. BHATT):—" The number $(10^p - 1)/9$ has one factor at least of the form $(ap + 1)$, where p and a are positive integers." Examine whether this theorem holds generally or no, and in the latter case find the least value of p for which it fails.

1413. (S. D. CHOWLA):—

If E_n is the nth Eulerian number, and B_n the nth *prepared* Bernoullian, prove that:

(i) $E_{n+2} - 187\,E_n + 126\,E_{n-1} \equiv 0 \pmod{576}$

(ii) $B_n - 20\,B_{n-1} + 24\,B_{n-2} \equiv 0 \pmod{2880}$.

1414. (S. AUDINARAYANAN, B. A. Hon.):—Evaluate

$$\int_0^\infty \frac{x^2}{\sinh^2 x}\,dx.$$

1415. (S. AUDINARAYANAN, B.A. Hon.).—If $\sinh x \sinh y = 1$, show that

$$\int_0^\infty y\,dx = \frac{\pi^2}{4}.$$

1416. (V. RAMASWAMI AIYAR):—If the tangent at a point P of a Cartesian Oval pass through the triple focus, then P is a point of inflexion on the confocal Cartesian passing through it.

1417. (V. RAMASWAMI AIYAR):—The tangents drawn to a three-cusped hypocycloid from a point P cut any tangent chord OO′ of the curve in F_1, F_2, F_3. Show that the Cartesian Ovals of which F_1, F_2, F_3 are single foci, and whose triple foci are O, O′ respectively, will intersect at P; and that the normals to them at P will be parallel to the tangents to the tricusp at O, O′ respectively.

1418. (S. D. CHOWLA):—Evaluate, in terms of Gamma Functions, the infinite product:—

$$\frac{5^{\frac{5}{5^2-1}} \cdot 17^{\frac{17}{17^2-1}} \cdot 29^{\frac{29}{29^2-1}} \cdot 41^{\frac{41}{41^2-1}} \cdots}{7^{\frac{7}{7^2-1}} \cdot 19^{\frac{19}{19^2-1}} \cdot 31^{\frac{31}{31^2-1}} \cdot 43^{\frac{43}{43^2-1}} \cdot 67^{\frac{67}{67^2-1}} \cdots}$$

where the numerator involves *all* primes of the form $12n + 5$, and the denominator all primes of the form $12n + 7$.

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during the Months of November and December 1925.

1 Acta Mathematica, **46**, 1 and 2.
2 American Journal of Mathematics, **47**, 3 and 4* (2 copies).
3 American Mathematical Monthly, **32**, 8 and 9 (2 copies).
4 Annals of Mathematics, **27**. 1 [and following back numbers **3**, 3; **14**, 4; **15**, 1 to 4; **16**, 1 to 4; **17**, 1; **18**, 1, 2, 4; **19**, 4].
5 Annales Scientifiques de l'ecole Normale Superieure, Juillet to Octobre 1925.
6 Astrophysical Journal, **62**, 3.
7 Bulletin des Sciences Math., **59**, Novem.
8 Contribucion al Estudio de la Ciencias, No. 67, Agosto 1925.
9 Japanese Journal of Math., **2**, 2.
10 Journal fur die reine und angewandte Mathematik, **154**, 4.
11 Mathematische Annalen, **95**, 2.
12 Memorial des Sciences Math. Fasc, 4 of 1925. Gauthier Vellars, Nomographic, by D'Ocagne.
13 Monthly Notices of the Royal Astronomical Society, **85**, 9; also the following: **31**, 2, 3, 4; **35**, 1, 2; **44**, 1, 2; **43**, 1, 2, 3; **42**, 2, 4.
14 Nature, **116**, 2919 to 2925.
15 Philosphical Magazine, Nos. 128, 217, 230, 232, 251, 252.
16 Philosophical Magazine, **50**, 299 and 300.
17 Philosophical Transactions of the Royal Society of London, **225**, 629, 630 and 631.
18 Proc. of the Edinburgh Math. Society, **43**, 2.
19 Proc. of the London Math. Society, **24**, 6 (3 copies).
20 Proc. of the Royal Society of London, **109**, A-751.
21 Popular Astronomy, **33**, 9 (2 copies).
22 Publications de la faculte des Sciences de L'Universite Masaryk:
  (1) Sur les Probabalites geometriques Par. B. Hostinsky.
  (2) Vyklad Vzniku etc. Napsal Dr. Josef Sahs.
  (3) Prispevek ke etc. Napsali J. Frejka, A. K. Vsetecka
  (4) O Absorpei Plynneho etc. Napsal Vaclai Cupr.
  (5) Novyzjev etc. Napsali A. Simek, A. H. Kadleova.
23 Quarterly Journal of Math., **50**, 3.
24 Tohoku Mathematical Journal, **26** 1 and 2.
25 Transactions of the American Math. Society, **27**, 4.

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